A PROOF OF ORE'S THEOREM

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ABSTRACT

The classical construction of the rational numbers involves consideration of certain equivalence classes of ordered pairs [(a,b)] where a and b are integers with b nonzero. An elementary generalization of this idea is Ore's Theorem which gives a necessary and sufficient condition that a ring, not necessarily commutative and not necessarily a domain of integrity, can be extended to a ring of "fractions." The purpose of this thesis is to analyze another proof of Ore's Theorem which involves a bare minimum of technique using the method of maximal extensions of semi-endomorphisms defined on a certain class of right ideals, i.e., given a ring with Ore's Condition we will construct the classical ring of right quotients.

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1 INTRODUCTION

As an algebraic system, the set of integers has the limitation that not every nonzero element has a multiplicative inverse. The importance of constructing a larger system in which every nonzero element is invertible is therefore recognized, i.e., the extension of the ring of integers to the field of rational fractions is certainly a natural problem to consider.

The classical construction of the rational numbers involves consideration of certain equivalence classes of ordered pairs [(a,b)] where a and b are integers with b nonzero. The equivalence class [(a,b)] thus corresponds to the usual concept of the rational number as the quotient $\frac{a}{b}$. It is immediately realized that this construction yields an extension field for any integral domain. An elementary generalization of this idea is Ore's Theorem which gives a necessary and sufficient condition that a ring, not necessarily commutative and not necessarily a domain of integrity, can be extended to a ring of "fractions." The classical proof of Ore's Theorem uses the same ordered pair construction employed to obtain the rational numbers.

The purpose of this thesis is to analyze another proof of Ore's Theorem which involves a bare minimum of technique using the method of maximal extensions of semi-endomorphisms (see [1],[4]) defined on a certain class of right ideals, i.e., given a ring with Ore's Condition we will construct the classical ring of right quotients. Other non-classical proofs are known (e.g. see [5], chapter 4, section 6). The motivation for this construction is again found in developing the rational numbers from the integers. As we generalize to any integral domain the maximal semi-endomorphisms can still be found, although less explicitly, through the use of Zorn's Lemma.

2 METHODS OF CONSTRUCTION

2.1 Classical Construction of the Rational Numbers

The classical construction of the field of rational numbers from the ring of integers considers the set of all ordered pairs (a,b) where a and b are integers with b nonzero. A relation is then defined on this set of ordered pairs by $(a,b) \sim (c,d)$ if and only if $ad = bc$. This relation is readily seen to be an equivalence relation. The set of equivalence classes becomes a field if we define addition and multiplication as follows. Addition is defined by $[(a,b)] + [(c,d)] = [(ad + bc, bd)]$. The associative and commutative properties are easily verified, $[(0,b)]$ is the zero, and $[(-a,b)]$ is the negative of $[(a,b)]$. The set of equivalence classes is thus an abelian group under addition. Multiplication is defined by $[(a,b)][(c,d)] = [(ac,bd)]$. The associative and commutative properties hold, the multiplicative identity is $[(b,b)]$, $b \neq 0$, and $[(b,a)]$ is the inverse for nonzero elements $[(a,b)]$, i.e., for elements $[(a,b)]$ such that $a \neq 0$. The distributive property is easily verified, and so the set of equivalence classes is a field. The integers are seen to be isomorphically contained in this field by defining a mapping ϕ from the integers into the set of equivalence classes of ordered pairs by $\phi(a) = [(ab,b)]$ for every integer a. ϕ may be verified to be an isomorphism. Observing that $[(a,b)] = [(ac,c)][(c,cb)] = [(ac,c)][(cb,c)]^{-1}$, $c \neq 0$, we can justify our writing the elements of this field as ab^{-1} or $\frac{a}{b}$ instead of $[(a,b)]$. Hence, we have the classical construction of the rational numbers from the integers.

2.2 Classical Proof of Ore's Theorem

In the above construction, the well-ordering property of the positive integers is never used, only the ring properties, commutativity, and the absence of zero divisors. In fact, the identity is never utilized. Thus, the construction yields a field of fractions for any integral domain D, i.e., an extension field Q(D) containing D isomorphically with the properties that every nonzero element of D is invertible in Q(D) and every element in Q(D) is of the form "$\frac{a}{b}$" $= ab^{-1} = b^{-1}a$ where $a,b \in D$, $b \neq 0$.

An elementary generalization of this idea is Ore's Theorem which replaces the commutativity assumption with a weaker property and allows nonzero divisors of zero in the ring. More precisely, we say a ring R satisfies Ore's Condition if for any $a,b \in R$, b regular, there exist $a_1,b_1 \in R$, b_1 regular, such that $ab_1 = ba_1$. A ring Q(R) containing R isomorphically is called a classical right quotient ring of R if every regular element of R is invertible in Q(R) and every element of Q(R) is of the form ab^{-1} where $a,b \in R$, b regular. Ore's Theorem is then: a ring R has a classical right quotient ring if and only if R satisfies Ore's Condition.

The necessity of Ore's Condition is seen immediately for if $a,b \in R$, b regular, then $b^{-1}a \in Q(R)$. Since every element in Q(R) is of the form xy^{-1}, $x,y \in R$, y regular, we have $b^{-1}a = a_1b_1^{-1}$ and hence $ab_1 = ba_1$.

The classical demonstration of the sufficiency of Ore's Condition is to mimic the ordered pair construction of the rational numbers. Explicitly, we consider $R \times R^*$ where R^* denotes the regular elements of R. Pairs in $R \times R^*$ are identified by $(a,b) \sim (c,d)$ if $ad_1 = cb_1$ where $db_1 = bd_1$ and b_1 is regular. (It follows that d_1 must also be regular.) This relation on ordered pairs is an equivalence relation and the set of equivalence classes a/b becomes a field when we define $a/b + c/d = (ad_1 + cb_1)/db_1$ with $bd_1 = db_1$ where $b_1, d_1 \varepsilon R^*$, and $(a/b)(c/d) = (ca_1)/(bg_1)$ with $ag_1 = da_1$ where $g_1 \varepsilon R^*$. The embedding of R in Q(R) is given by identifying $a \varepsilon R$ with $ab_1/b_1 \varepsilon Q(R)$, b_1 regular. The arguments used to show that this construction is valid essentially use Ore's Condition in those places where commutativity was needed in the construction for an integral domain. The arguments which depended on cancellation (non-zero divisors) only involve the second coordinate of pairs (a,b), and in this case b is always regular.

2.3 Non-classical Construction of the Rational Numbers

A second look at the ring of integers Z and the field of rational fractions from a different point of view suggests another method of constructing extensions (as introduced in [1]). We assume an elementary idea about the ideal structure of the ring of integers; namely, the ring of integers is a principal ideal domain, i.e., every ideal is of the form $(m) = \{x \mid x = mk, k \varepsilon Z\}$ where $m \varepsilon Z$.

Given a homomorphism $f : (m) \to Z$, it is completely determined by its value at m, i.e., if $f(m) = n$, and $t \varepsilon (m)$, then $t = mk$ and $f(t) = f(mk) = f(m)k = nk$. Furthermore,

$f(t) = f(m)k = \frac{f(m)}{m}(mk) = \frac{f(m)}{m}t$, i.e., f is determined by the fraction $\frac{f(m)}{m}$. For example, if $f:(154)\to Z$ with $f(154)=30$, then $f(t) = \frac{30}{154}t$ for every $t \in (154)$. This suggests identifying a fraction with a "semi-endomorphism" (i.e., a homomorphism from an ideal of Z into Z) by "$\frac{a}{b}$" corresponds to $f_a^b:(b)\to Z$ with $f(b) = a$.

We desire to identify certain fractions. For example, $\frac{60}{308}$ is to be equivalent to $\frac{30}{154}$. Note that $f_{60}^{308} = f_{30}^{154}$ on $(308) \cap (154)$. Furthermore, we wish to "reduce fractions to lowest terms," i.e., we identify both $\frac{60}{308}$ and $\frac{30}{154}$ with $\frac{15}{77}$. Note that f_{15}^{77} is an extension of both f_{60}^{308} and f_{30}^{154}. Also f_{15}^{77} is a maximal extension of each for there is no extension of f_{15}^{77} to an ideal containing (77). These observations suggest the following ideas.

An equivalence relation on the set of semi-endomorphisms is given by $f_a^b \sim f_c^d$ if and only if $f_a^b = f_c^d$ on $(b) \cap (d)$, i.e., if and only if $ad = bc$. We say f_a^b is maximal if it cannot be extended to $(m)\supsetneq(b)$. Note that f_a^b is maximal if a and b are relatively prime. If G.C.D.$\{a,b\} = d > 1$, then f_a^b can be extended to $\widehat{f_a^b} : (\frac{b}{d})\to Z$ where $\widehat{f_a^b}(\frac{b}{d}) = \frac{a}{d}$. And $\widehat{f_a^b} = f_{a'}^{b'}$, where $b' = \frac{b}{d}$ and $a' = \frac{a}{d}$, is maximal since G.C.D.$\{a',b'\} = 1$. In other words, every semi-endomorphism has a unique maximal extension. Also, $f_a^b \sim f_c^d$ if and only if $\widehat{f_a^b} = \widehat{f_c^d}$.

Addition of equivalence classes is defined by $[f_a^b] + [f_c^d] = [f_a^b + f_c^d]$ where $f_a^b + f_c^d : (b) \cap (d)\to Z$ by $f_a^b + f_c^d(x) = f_a^b(x) + f_c^d(x)$. It is readily seen that $f_a^b + f_c^d \sim f_{au+cv}^m$ where m is the least common multiple of b and d, and

$bu = m = dv$. This yields a commutative group with zero $[f_0^b]$; the negative of $[f_a^b]$ is given by $[f_{-a}^b]$.

Multiplication is defined by $[f_a^b][f_c^d] = [f_a^b \circ f_c^d]$ where $f_a^b \circ f_c^d : (bd) \to Z$ by composition. It is easily shown that $f_a^b \circ f_c^d \sim f_{ac}^{bd}$. This yields a semi-group with identity $[f_b^b]$, $b \neq 0$, and with $[f_b^a]$ acting as the inverse of $[f_a^b]$, $b \neq 0$. In addition, $f_a^b \sim f_{ac}^c \circ f_c^{bc}$, $c \neq 0$. The distributive law holds and thus we get a field. Z is established isomorphically as a subring of this field by the correspondence $a \to f_{ab}^b$, $b \neq 0$.

This construction uses the well-ordering property of the integers (contrary to the ordered pair construction). Thus, in attempting to apply this method to more general classes of rings, e.g., integral domains or rings with Ore's Condition, we will make use of Zorn's Lemma to establish the existence of maximal extensions of semi-endomorphisms.

3 A PROOF OF ORE'S THEOREM

In this chapter we give a constructive proof of Ore's Theorem using the idea of maximal extensions of semi-endomorphisms defined on a certain class of right ideals, i.e., given a ring with Ore's Condition, we will construct the classical ring of right quotients.

Definition 1: A ring R satisfies Ore's Condition if and only if for every $a,b \in R$ with b regular, there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$.

Let R be a ring satisfying Ore's Condition with $R^* = \{$regular elements of $R\} \neq \{0\}$.
(An element x is regular if there is no $y \neq 0$ with $yx=0$ or $xy=0$.)

Let $M = \{I \mid I$ is a right ideal of R containing at least one regular element$\}$.

Definition 2: A semi-endomorphism is a mapping $f:I_f \rightarrow R$ where $I_f \in M$ such that $f(x + y) = f(x) + f(y)$ and $f(xr) = f(x)r$ for $x,y \in I_f$, $r \in R$.

Let $H = \{f \mid f$ is a semi-endomorphism $f:I_f \rightarrow R$, $I_f \in M\}$.

Definition 3: For $f,g \in H$, define $f \leq g$ if and only if $I_f \subseteq I_g$ and $f(x) = g(x)$ for every $x \in I_f$.

Proposition 1: $\leq$ is a partial ordering on H.

Proof: (i) $I_f \subseteq I_f$ and $f(x) = f(x)$ for every $x \in I_f$, so $f \leq f$

(ii) If $f \leq g$ and $g \leq f$, then $I_f \subseteq I_g$ and $I_g \subseteq I_f$ so $I_f = I_g$;

and $f(x) = g(x)$ for every $x \in I_f$, $g(x) = f(x)$ for every $x \in I_g$, so $f(x) = g(x)$ for every $x \in I_f = I_g$. Thus, $f = g$.

(iii) If $f \leq g$ and $g \leq h$, then $I_f \subseteq I_g$ and $I_g \subseteq I_h$, so $I_f \subseteq I_h$; and $f(x) = g(x)$ for every $x \in I_f$, $g(x) = h(x)$ for every $x \in I_g$, so $f(x) = h(x)$ for every $x \in I_f$. Thus, $f \leq h$.

Therefore, $(H, \leq)$ is a partially ordered set.

<u>Proposition 2</u>: If $f \in H$, f has a maximal extension, i.e., there exists $\hat{f} \in H$, $\hat{f} \geq f$ and if $g \in H$, $g \geq \hat{f}$, then $g = \hat{f}$.

Proof: Apply Zorn's Lemma, i.e., show every chain in H has an upper bound. Let S be a totally ordered subset of H. Let $I = \bigcup I_\beta$, $\beta \in S$. Obviously $I \in M$. Let $\alpha: I \rightarrow R$ be defined by $\alpha(x) = \beta(x)$ whenever $x \in I_\beta$, $\beta \in S$. Let $f \in S$ such that $f: I_f \rightarrow R$. Now $I_f \subseteq I$ and $f(x) = \alpha(x)$ for every $x \in I_f$. Hence, $f \leq \alpha$. Thus, $f \leq \alpha$ for all $f \in S$, i.e., S has an upper bound. Therefore, H is a partially ordered set such that every chain in H has an upper bound. So by Zorn's Lemma, H contains a maximal element. That is, each semi-endomorphism has a maximal extension.

<u>Proposition 3</u>: If $I_f, I_g \in M$, then $I_f \cap I_g \in M$.

Proof: Clearly $I_f \cap I_g$ is a right ideal since I_f and I_g are right ideals. Now, to show $I_f \cap I_g$ contains at least one regular element. Let $a \in I_f$, a regular. Let $b \in I_g$, b regular. Then by Ore's Condition there exist $a_1, b_1 \in R$,

b_1 regular, such that $ab_1 = ba_1$. Now, $ab_1 \varepsilon I_f$ since $b_1 \varepsilon R$ and I_f is a right ideal. Also, $ba_1 \varepsilon I_g$ since $a_1 \varepsilon R$ and I_g is a right ideal. But $ab_1 = ba_1$ so let $x = ab_1 = ba_1$ and we have $x \varepsilon I_f \cap I_g$. And since a is regular and b_1 is regular, then clearly ab_1 is regular. So x is regular. Thus, $I_f \cap I_g$ has at least one regular element.

Lemma 4: If $f,g \varepsilon H$ and $f = g$ on $I \subseteq I_f \cap I_g$, $I \varepsilon M$, then f and g have a common extension k such that $k:I_f+I_g \to R$.

Proof: Let $x \varepsilon I_f \cap I_g$, $x \not\varepsilon I$. Let $y \varepsilon I$, y regular. Suppose $f(x) \neq g(x)$. By Ore's Condition there exist $x_1, y_1 \varepsilon R$, y_1 regular, such that $xy_1 = yx_1$. Now $f(x) - g(x) \neq 0$, so $0 \neq [f(x) - g(x)]y_1 = f(x)y_1 - g(x)y_1 = f(xy_1) - g(xy_1) = f(yx_1) - g(yx_1)$. But $yx_1 \varepsilon I$, and $f = g$ on I. Thus, we have a contradiction, and so $f = g$ on $I_f \cap I_g$.

Now let $x \varepsilon I_f$, $y \varepsilon I_g$. Then define $k:I_f+I_g \to R$ by $k(x + y) = f(x) + g(y)$. To show it is well-defined, suppose $x + y = x' + y'$, $x,x' \varepsilon I_f$ and $y,y' \varepsilon I_g$. Then $x - x' = y' - y \varepsilon I_f \cap I_g$, so $f(x - x') = g(y' - y)$. Thus, $f(x) - f(x') = g(y') - g(y)$ which implies that $f(x) + g(y) = f(x') + g(y')$. Hence, $k(x + y) = f(x) + g(y) = f(x') + g(y') = k(x' + y')$. Clearly $I_f + I_g \varepsilon M$ and k is a semi-endomorphism. Now $I_f \subseteq I_f + I_g$ and $f(x + y) = k(x + y)$ for every $x + y \varepsilon I_f$, so $f \leq k$. And $I_g \subseteq I_f + I_g$ and $g(x + y) = k(x + y)$ for every $x + y \varepsilon I_g$, so $g \leq k$. Hence, $k:I_f+I_g \to R$ is a common extension of f and g.

Corollary: If $f = g$ on $I_1 \varepsilon M$, then f and g have a common extension.

Proof: Let $I = I_1 \cap I_f \cap I_g \varepsilon M$. Then $f = g$ on I. So f and g have a common extension.

Proposition 5: If $f \varepsilon H$, then f has a unique maximal extension.

Proof: Let $f \varepsilon H$. Let f_1 and f_2 be maximal extensions of f. Now $I_f \subseteq I_{f_1} \cap I_{f_2}$ and $f_1 = f_2$ on I_f. Thus, by the previous lemma, f_1 and f_2 have a common extension k. But since f_1 and f_2 are maximal it must be that $f_1 = k = f_2$. Hence, f has a unique maximal extension.

Notation: For $f \varepsilon H$, $\hat{f}$ denotes its unique maximal extension.

Definition 4: For $f,g \varepsilon H$, $f \sim g$ if and only if f and g have the same maximal extension. $\sim$ is obviously an equivalence relation on H.

Theorem 6: $\hat{f} = \hat{g}$ if and only if $f = g$ on $I \varepsilon M$.

Proof: (i) Let $f = g$ on $I \varepsilon M$. Then f and g have a common extension k, so they have the same maximal extension. Thus, $\hat{f} = \hat{g}$.

(ii) Let $\hat{f} = \hat{g}$. Then there exists a maximal extension $\hat{f}_1$ such that $\hat{f} = \hat{f}_1$ and $\hat{g} = \hat{f}_1$. So $f = \hat{f}_1$ on I_f and $g = \hat{f}_1$ on I_g. Hence, on $I = I_f \cap I_g \varepsilon M$, $f = g$.

Definition 5: Let $\Lambda = \{\hat{f} \mid f \varepsilon H\}$.

Note: We could think of each element in Λ as an equivalence class, the equivalence classes being those sets of semi-endomorphisms in H which have the same maximal extension.

However, a canonical representative of each equivalence class is that unique maximal extension.

Definition 6: For $\hat{f},\hat{g} \in \Lambda$, define $\hat{f} + \hat{g}$ to be $\widehat{f+g}$ where $f+g: I_f \cap I_g \to R$ by $f+g(x) = f(x) + g(x)$.

Proposition 7: If $f \sim f'$ and $g \sim g'$, then $\widehat{f+g} = \widehat{f'+g'}$.

Proof: $f + g = f' + g'$ on $I_f \cap I_g \cap I_{f'} \cap I_{g'}$, so $\widehat{f+g} = \widehat{f'+g'}$ by Theorem 6.

Proposition 8: Addition is commutative, i.e., if $\hat{f},\hat{g} \in \Lambda$ then $\hat{f} + \hat{g} = \hat{g} + \hat{f}$.

Proof: Let $\hat{f},\hat{g} \in \Lambda$. Then $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$ for every $x \in I_f \cap I_g$. So $\widehat{f+g} = \widehat{g+f}$.

Proposition 9: Addition is associative, i.e., if $\hat{f},\hat{g},\hat{h} \in \Lambda$ then $\hat{f} + (\hat{g} + \hat{h}) = (\hat{f} + \hat{g}) + \hat{h}$.

Proof: Let $\hat{f},\hat{g},\hat{h} \in \Lambda$. Then $[f + (g + h)](x) = f(x) + (g + h)(x) = f(x) + [g(x) + h(x)] = [f(x) + g(x)] + h(x) = (f + g)(x) + h(x) = [(f + g) + h](x)$ for every $x \in I_f \cap I_g \cap I_h$. So $\widehat{f+(g+h)} = \widehat{(f+g)+h}$.

Proposition 10: There is an identity for addition, i.e., there exists a semi-endomorphism $O \in \Lambda$ such that $\hat{f} + O = O + \hat{f} = \hat{f}$ for every $\hat{f} \in \Lambda$.

Proof: Define $O: R \to R$ such that $O(x) = 0$ for every $x \in R$. Clearly $O \in \Lambda$. Let $\hat{f} \in \Lambda$, $\hat{f}: I_f \to R$. Then $(f + O)(x) = f(x) + O(x) = f(x) + 0 = f(x)$ for every $x \in I_f$ and $(O + f)(x) = O(x) + f(x) = 0 + f(x) = f(x)$ for every $x \in I_f$. So $\widehat{f+O} = \hat{f} = \widehat{O+f}$.

Proposition 11: There is an inverse for addition, i.e., there exists a semi-endomorphism $-\hat{f} \in \Lambda$ such that $\hat{f} + -\hat{f} = 0 = -\hat{f} + \hat{f}$ for $\hat{f} \in \Lambda$.

Proof: Let $\hat{f} \in \Lambda$, $\hat{f}: I_f \to R$. Define $-\hat{f}: I_f \to R$ by $(-f)(x) = -[f(x)]$ for every $x \in I_f$. Clearly $-\hat{f} \in \Lambda$ since $\hat{f} \in \Lambda$. Then $(f + -f)(x) = f(x) + (-f)(x) = f(x) + -[f(x)] = 0 = 0(x)$ for every $x \in I_f$. And $(-f + f)(x) = (-f)(x) + f(x) = -[f(x)] + f(x) = 0 = 0(x)$ for every $x \in I_f$. Therefore, $\widehat{f+-f} = 0 = \widehat{-f+f}$.

Hence, $(H,+)$ is an abelian group under addition.

Definition 7: For $\hat{f},\hat{g} \in \Lambda$ define $\hat{f}\hat{g}$ to be $\widehat{f \circ g}$ where $f \circ g: I_g^f \to R$, $I_g^f = \{x \mid x \in I_g \text{ and } g(x) \in I_f\}$, such that $f \circ g(x) = f(g(x))$ for every $x \in I_g^f$.

Note: $\widehat{f \circ g} = \widehat{\hat{f} \circ \hat{g}}$

Proposition 12: $I_g^f = \{x | x \in I_g \text{ and } g(x) \in I_f\} = I_g \cap g^{-1}(I_f) \in M$.

Proof: Let $x \in I_g$, x regular. Let $y \in I_f$, y regular. Then $g(x) \in R$. So by Ore's Condition there exist $a_1, b_1 \in R$, b_1 regular, such that $ya_1 = g(x)b_1$. Now $xb_1 \in I_g$ since $x \in I_g$ and I_g is a right ideal. And since x and b_1 are regular, then xb_1 is regular. Now, $g(xb_1) = g(x)b_1 = ya_1 \in I_f$ since $y \in I_f$. So $xb_1 \in I_g^f$ and xb_1 is regular. Thus, I_g^f contains at least one regular element. Now let $x,y \in I_g^f$. Then $x \in I_g$ and $y \in I_g$, so $x + y \in I_g$. And $g(x) \in I_f$, $g(y) \in I_f$, so $g(x) + g(y) = g(x + y) \in I_f$. Thus, $x + y \in I_g^f$.

Let $x \in I_g^f$. Then $x \in I_g$ which implies $-x \in I_g$. And $g(x) \in I_f$, so $-[g(x)] \in I_f$. Now $-[g(x)] + g(x) = 0$

implies $-[g(x)] + g(x) + g(-x) = 0 + g(-x)$ implies $-[g(x)] + g(x + -x) = 0 + g(-x)$ implies $-[g(x)] + g(0) = 0 + g(-x)$ implies $-[g(x)] + 0 = 0 + g(-x)$ implies $-[g(x)] = g(-x)$. Thus, since $-[g(x)] \in I_f$, then $g(-x) \in I_f$. Therefore, I_g^f is a subgroup of R under addition.

Now let $x \in I_g^f$ and $r \in R$. Then $x \in I_g$ and $g(x) \in I_f$. So $xr \in I_g$, and $g(x)r = g(xr) \in I_f$ since $g(x) \in I_f$. So $xr \in I_g^f$. Hence, I_g^f is a right ideal of R.

Proposition 13: Multiplication is associative, i.e., if $\hat{f}_1, \hat{f}_2, \hat{f}_3 \in \Lambda$ then $\hat{f}_1(\hat{f}_2\hat{f}_3) = (\hat{f}_1\hat{f}_2)\hat{f}_3$.

Proof: Let $I = \{x \in I_3 \mid f_3(x) \in I_2$ and $f_2(f_3(x)) \in I_1\}$. Then $I = I_3 \cap f_3^{-1}[(f_2^{-1}(I_1) \cap I_2] \cap I_3$. Now $I_4 = f_2^{-1}(I_1) \cap I_2 \in M$ by Proposition 12, so $I = I_3 \cap f_3^{-1}(I_4) \in M$ by Proposition 12. And $f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ f_3$ on $I \in M$. Therefore, the assertion follows by Theorem 6.

Proposition 14: Multiplication is distributive, i.e., for $\hat{f}_1, \hat{f}_2, \hat{f}_3 \in \Lambda$, $\hat{f}_1(\hat{f}_2 + \hat{f}_3) = \hat{f}_1\hat{f}_2 + \hat{f}_1\hat{f}_3$ and $(\hat{f}_2 + \hat{f}_3)\hat{f}_1 = \hat{f}_2\hat{f}_1 + \hat{f}_3\hat{f}_1$.

Proof: $[f_1 \circ (f_2 + f_3)](x) = f_1((f_2 + f_3)(x)) = f_1(f_2(x) + f_3(x)) = f_1(f_2(x)) + f_1(f_3(x)) = f_1 \circ f_2(x) + f_1 \circ f_3(x)$ on $I_2^1 \cap I_3^1 \in M$. So $\widehat{f_1 \circ (f_2 + f_3)} = \widehat{f_1 \circ f_2 + f_1 \circ f_3}$ by Theorem 6.

$[(f_2 + f_3) \circ f_1](x) = (f_2 + f_3)(f_1(x)) = f_2(f_1(x)) + f_3(f_1(x)) = f_2 \circ f_1(x) + f_3 \circ f_1(x)$ on $I_1^2 \cap I_1^3 \in M$. So $\widehat{(f_2 + f_3) \circ f_1} = \widehat{f_2 \circ f_1 + f_3 \circ f_1}$ by Theorem 6.

Thus, we have shown the following theorem.

Theorem 15: Λ is a ring.

Proposition 16: If $f \neq 0$, then $f(a) \neq 0$ for every regular element a.

Proof: Let $f \in \Lambda$. Let $f(a) = 0$ for some regular element $a \in I_f$. Suppose there exists an element $b \in I_f$ such that $f(b) \neq 0$. Now by Ore's Condition, there exist $b_1, a_1 \in R$, a_1 regular, such that $ab_1 = ba_1$. Then $0 \neq f(b)a_1 = f(ba_1) = f(ab_1) = f(a)b_1$. But $f(a)b_1 = 0$ since $f(a) = 0$. Thus we have a contradiction, and so $f(b) = 0$. Therefore, $f = 0$.

Definition 8: Define $f_a\colon R \to R$ by $f_a(x) = ax$ where $a \in R$.

Note: $f_a = \hat{f}_a \in \Lambda$

Definition 9: $R' = \{f_a | a \in R\}$

Definition 10: Define $\psi\colon R \to R'$ by $\psi(a) = f_a$.

Proposition 17: R' is a subring of Λ isomorphic to R.

Proof: Show ψ is an isomorphism. Let $a, b \in R$. Now $f_{a+b}(x) = (a+b)x = ax + bx = f_a(x) + f_b(x)$. So $\psi(a+b) = f_{a+b} = f_a + f_b = \psi(a) + \psi(b)$. Also, $f_{ab}(x) = (ab)x = a(bx)$ and $f_a f_b(x) = f_a(f_b(x)) = f_a(bx) = a(bx)$, so $f_{ab} = f_a f_b$. Thus, $\psi(ab) = f_{ab} = f_a f_b = \psi(a)\psi(b)$. So ψ is a homomorphism.

Now if $f_a = f_b$ then $ax = bx$ for every $x \in R$. So there exists an x_o, x_o regular, such that $ax_o = bx_o$ which implies $ax_o - bx_o = 0$ which implies $(a-b)x_o = 0$ which implies

$a - b = 0$ which implies $a = b$. So ψ is one-to-one. Clearly ψ is onto. Therefore, ψ is an isomorphism.

Proposition 18: If $a \in R^*$ then $\hat{f}_a^{-1}$ exists.

Proof: f_a is a one-to-one function since $f_a(x) = f_a(y)$ implies $ax = ay$ implies $x = y$ since a is regular. Thus, f_a^{-1} exists. Now $a^2 \in$ Image f_a, and a^2 is regular since a is regular. Therefore, Image $f_a \in M$. We have $f_a \circ f_a^{-1} = i$ on Image f_a where i is the identity mapping. So $\hat{f}_a \widehat{f_a^{-1}} = \widehat{f_a \circ f_a^{-1}} = \hat{i}$. And $f_a^{-1} \circ f_a = i$ on R, so $\widehat{f_a^{-1}} \hat{f}_a = \widehat{f_a^{-1} \circ f_a} = \hat{i}$. Therefore, $\widehat{f_a^{-1}} = \hat{f}_a^{-1}$.

Therefore, Λ contains R isomorphically, and regular elements of R are invertible in Λ.

Proposition 19: If $\hat{f} \in \Lambda$, then $\hat{f} = \hat{f}_a \widehat{f_b^{-1}}$ where $a,b \in R$ with b regular.

Proof: Let $\hat{f} \in \Lambda$. Let $b \in I_f$, b regular. Then $bR \subseteq I_f$. Let $x \in R$. Then $f(bx) = f(b)x$. Let $f(b) = a \in R$. Then $f \circ f_b(x) = f(f_b(x)) = f(bx) = f(b)x = ax = f_a(x)$. So $f \circ f_b = f_a$. And since $b \in R$, b regular, f_b^{-1} exists. So $f = f_a \circ f_b^{-1}$ on I_f. Therefore, $\hat{f} = \widehat{f_a \circ f_b^{-1}} = \hat{f}_a \widehat{f_b^{-1}}$.

Thus, we have constructed a classical right quotient ring containing R.

We know there is an isomorphism between Λ and $Q(R)$, the right quotient ring obtained by the classical construction; namely, the correspondence "ab^{-1}"$\to$"ab^{-1}." However, we give below another isomorphism which reveals more explicitly the correspondence between functions in Λ and classes of ordered pairs in $Q(R)$.

Define $\phi: \Lambda \to Q(R)$ by $\phi(\hat{f}) = [(f(a),a)]$ where $a \in I_f \cap R^*$. Recall that $(a,b) \sim (c,d)$ means $ad_1 = cb_1$ where $db_1 = bd_1$, b_1 and d_1 regular.

First we show ϕ is independent of which regular element in the domain of f is chosen. Let $b,d \in I_f$, b and d regular. Then by Ore's Condition there exist $b_1,d_1 \in R$, d_1 regular (and hence b_1 regular) such that $bd_1 = db_1$. So $f(bd_1) = f(db_1)$ which implies $f(b)d_1 = f(d)b_1$ which implies $(f(b),b) \sim (f(d),d)$. So $[(f(b),b)] = [(f(d),d)]$.

Now we show ϕ is one-to-one. Let $\hat{f},\hat{g} \in \Lambda$. Let $a \in I_f \cap I_g$, a regular. Suppose $[(f(a),a)] = [(g(a),a)]$. Then $(f(a),a) \sim (g(a),a)$. So $f(a)d_1 = g(a)b_1$ where $ab_1 = ad_1$, b_1,d_1 regular. But $ab_1 = ad_1$ implies $ab_1 - ad_1 = 0$ implies $a(b_1 - d_1) = 0$ implies $b_1 - d_1 = 0$ implies $b_1 = d_1$. Thus, $f(a)d_1 = g(a)b_1$ implies $f(a)d_1 = g(a)d_1$ implies $f(a)d_1 - g(a)d_1 = 0$ implies $(f(a) - g(a))d_1 = 0$ implies $f(a) - g(a) = 0$ implies $f(a) = g(a)$. That is, $f(a) = g(a)$ for every $a \in I_f \cap I_g$, a regular. Let $b \in I_f \cap I_g$. By Ore's Condition there exist $a_1,b_1 \in R$, a_1 regular, such that $ba_1 = ab_1$. Suppose $f(b) \neq g(b)$. Then $f(b) - g(b) \neq 0$. So $0 \neq [f(b) - g(b)]a_1 = f(b)a_1 - g(b)a_1 = f(ba_1) - g(ba_1) = f(ab_1) - g(ab_1)$. But $ab_1 \in I_f \cap I_g$, and ab_1 is regular because for any $z \in R$, $(ab_1)z = a(b_1z) \neq 0$ since a is regular. Thus, $f(ab_1) - g(ab_1) \neq 0$ is a contradiction. So $f(b) = g(b)$ for all $b \in I_f \cap I_g$. Therefore, $\phi(\hat{f}) = \phi(\hat{g})$ implies $\hat{f} = \hat{g}$, so ϕ is one-to-one.

ϕ is onto, for let $[(x,y)] \varepsilon Q(R)$. Then $x,y \varepsilon R$, y regular. Now there exists an element $c \varepsilon Q(R)$ such that $x = cy$. [Note: c may be thought of as xy^{-1}] Define $f{:}I \to R$ by $f(t) = ct$ where $I = yR$. Now, $I \varepsilon M$ since I is a right ideal of R and I contains the regular element y. For $t \varepsilon I$, $t = yr$ for some $r \varepsilon R$, so $f(t) = cyr = xr \varepsilon R$. Thus, f maps an ideal of M into R. Now f is a semi-endomorphism since for $a,b \varepsilon I$, $r \varepsilon R$, we have $f(a + b) = c(a + b) = ca + cb = f(a) + f(b)$, and $f(ar) = c(ar) = (ca)r = f(a)r$. Therefore, $y \varepsilon I$ and $f(y) = cy = x$, so $[(x,y)] = [(f(y),y)] = \phi(\hat{f})$. Hence, ϕ is onto.

ϕ is also a homomorphism, and thus ϕ is an isomorphism from the semi-endomorphisms in Λ to classes of ordered pairs in $Q(R)$.

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